

Performance Analysis of Adaptive MIMO OFDM Beamforming Systems

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Abstract—In this paper we consider an adaptive modulation system with multiple-input multiple-output (MIMO) antennas in conjunction with orthogonal frequency division multiplexing (OFDM) operating over frequency selective Rayleigh fading environments. In particular, we consider a type of beamforming with a maximum ratio transmission, maximum ratio combining (MRT-MRC) transceiver structure. For this system we derive a central limit theorem for various block-based performance metrics. This motivates an accurate Gaussian approximation to the system data rate and the number of outages per OFDM block. In addition to data rate and outage distributions, we also consider the subcarrier SNR as a random process in the frequency domain and compute level crossing rates (LCRs) and average fade bandwidths (AFBs). Hence, we provide fundamental but novel results for the MIMO OFDM channel. The accuracy of these results is verified by Monte Carlo simulations, and applications to both performance analysis and system design are discussed.

I. INTRODUCTION

MIMO OFDM systems, involving clever processing in both the spatial and frequency domains, have been proposed for WiFi, WiMax and fourth generation cellular systems as well as the IEEE 802.16 standard for wireless Internet access. To enhance the system throughput, adaptive modulation schemes for MIMO OFDM systems have recently been developed. Our work focuses on the statistical analysis of adaptive MIMO OFDM beamforming systems, providing important insights into the system throughput and outage probability. In particular, we derive closed-form approximations for the cumulative distribution function (CDF) and the exact mean and variance of the number of bits transmitted per OFDM block and the number of outages per OFDM block. In addition, we analyze the level crossing rate (LCR) and average fade bandwidth (AFB) of the subcarrier SNR across frequency. These are important statistics for designing error control codes, channel estimation procedures and diversity schemes for mobile communication systems.

The statistical variation of various metrics across subcarriers in OFDM is a fundamental problem in OFDM performance, but one that has received little attention, probably due to the mathematical challenges involved. Hence the focus of our work is the characterization of MIMO OFDM systems across frequency. In addition to considering data rate and outage distributions we also consider the subcarrier SNR as a random process in the frequency domain and compute level

crossing rates and average fade bandwidths. Hence, we provide fundamental but novel results for the MIMO OFDM channel.

Furthermore, our analysis can also be used as a framework for other studies where the behavior over frequency is important. For example, similar approaches in [1] and [2] have led to results on the capacity of MIMO OFDM systems and the BER of SISO OFDM systems. Applications of this work can be found in both performance analysis and design. The data rate and outage distributions give more complete performance results than previously available.

II. MIMO OFDM SYSTEM

We consider an adaptive MIMO OFDM beamforming system transmitting over N subcarriers with N_T antennas at the transmitter and N_R antennas at the receiver. The system transmits data symbol S_k on the k -th subcarrier for $k \in \{1, 2, \dots, N\}$, where $S_k \in \mathbb{R}^2$ is from some two-dimensional symbol constellation. We refer to the superposition of all N modulated subcarriers as the OFDM block. We assume that each subcarrier occupies a subchannel of bandwidth Δf (Hz), yielding a total bandwidth of $B = N \Delta f$. Furthermore, each subcarrier symbol is transmitted with equal energy E_s such that the total average transmitted energy is $E_N = N E_s$.

At the transmitter, the k -th subcarrier modulates the symbol S_k using the beamforming vector (or weight vector) \mathbf{b}_k . We assume that the sampled impulse response of the channel is shorter than the cyclic prefix. After removing the cyclic prefix, the channel for the k -th subcarrier after the Discrete Fourier Transform (DFT) can then be described as a $N_R \times N_T$ complex channel matrix \mathbf{H}_k . Considering a beamforming-combining system, the output of the combiner at the receiver on the k -th subcarrier can be written as

$$R_k = \mathbf{z}_k^\dagger \mathbf{H}_k \mathbf{b}_k S_k + \mathbf{z}_k^\dagger \mathbf{n}_k \quad (1)$$

where \dagger represents the conjugate transpose, \mathbf{z}_k is the combiner weight vector and \mathbf{H}_k is the narrowband channel transfer function for subcarrier k . The noise vector is denoted by \mathbf{n}_k with independent and identically distributed (i.i.d.) Gaussian entries distributed according to $\mathcal{CN}(0, \sigma^2)$. We set $\|\mathbf{b}_k\| = 1$ to reflect the power constraint at the transmitter, where $\|\cdot\|$ denotes the Euclidian norm.

For a given beamforming vector \mathbf{b}_k , the combining vector

z_k that maximizes the SNR is given by [3]

$$z_k = \frac{\mathbf{H}_k \mathbf{b}_k}{\|\mathbf{H}_k \mathbf{b}_k\|}. \quad (2)$$

Then, varying the beamforming vector, the maximum SNR is achieved if \mathbf{b}_k is proportional to the eigenvector corresponding to the maximum eigenvalue $\lambda_{\max}^{(k)}$ of $\mathbf{H}_k \mathbf{H}_k^\dagger$. This transmission scheme is commonly described as maximum ratio transmission and maximum ratio combining (MRT-MRC), which achieves full diversity and the full array gain in Rayleigh fading channels [4], [3]. Substituting this eigenvector into (2), the resulting optimal SNR can be written as

$$\gamma_{\max}^{(k)} = \frac{E_s}{\sigma^2} \lambda_{\max}^{(k)}, \quad (3)$$

where $\frac{E_s}{\sigma^2}$ denotes the average SNR per branch, and (1) can be replaced by

$$R_k = \sqrt{\lambda_{\max}^{(k)}} S_k + \tilde{n}, \quad (4)$$

where $\tilde{n} \sim \mathcal{CN}(0, \sigma^2)$ is a complex Gaussian noise term independent of $\lambda_{\max}^{(k)}$. This simple formulation of the received signal is a necessary result for the forthcoming modulation switching threshold calculation. From (3) we see that the subcarrier SNR, $\gamma_{\max}^{(k)}$ is proportional to $\lambda_{\max}^{(k)}$. Hence, the adaptive system can select the modulation scheme based on the maximum eigenvalue.

A. Frequency Selective Channel

We assume a familiarity with frequency selective Rayleigh fading channels and use the well-known Jakes' model [5]. We make the general assumption of a frequency selective Rayleigh fading channel that is wide sense stationary with uncorrelated, isotropic scattering. Furthermore, we presume that the delay autocorrelation function may be described as an exponential delay power profile with rms delay τ_d . However, note that the analysis developed later does not depend on the type of delay power profile. We select an arbitrary time point and only consider variation across frequency. The subchannel gains of the k_1 -th and k_2 -th subcarriers can be written as

$$H_{k_1} = X_{k_1} + j Y_{k_1} \quad \text{and} \quad H_{k_2} = X_{k_2} + j Y_{k_2} \quad (5)$$

where X_{k_1} , Y_{k_1} , X_{k_2} and Y_{k_2} are identically distributed zero mean Gaussian random variables. Without loss of generality we may set $E[X_k^2] = E[Y_k^2] = \frac{1}{2}$, for all k . Following [5], we may then write the cross-correlations

$$\begin{aligned} E[X_{k_1} X_{k_2}] &= E[Y_{k_1} Y_{k_2}] = \frac{1}{2} \frac{1}{1 + (2\pi \tau_d \Delta f \Delta k)^2} \\ E[X_{k_1} Y_{k_1}] &= E[X_{k_2} Y_{k_2}] = 0 \\ E[X_{k_1} Y_{k_2}] &= -E[X_{k_2} Y_{k_1}] = -(2\pi \tau_d \Delta f \Delta k) E[X_{k_1} X_{k_2}] \end{aligned} \quad (6)$$

where $\Delta k = |k_1 - k_2|$. With these definitions we obtain the correlation function

$$\rho_f(\Delta k \Delta f) = E[H_{k_1} H_{k_2}^*] = \frac{1 + j 2\pi \tau_d \Delta f \Delta k}{1 + (2\pi \tau_d \Delta f \Delta k)^2}. \quad (7)$$

Note that from (5) the marginal distribution of each channel gain $|H_k|^2$ follows an exponential distribution with $E[|H_k|^2] = 1$, $\text{var}[|H_k|^2] = 1$ and

$$\text{corr}(|H_k|^2, |H_{k+\Delta k}|^2) = \frac{1}{1 + (2\pi \tau_d \Delta f \Delta k)^2} \quad (8)$$

where $\text{corr}(\cdot, \cdot)$ represents the correlation coefficient.

In this paper, we consider a MIMO system with independent channel coefficients in the $N_R \times N_T$ channel matrix, \mathbf{H}_k , for all subcarriers k . This is a reasonable assumption in urban environments or when the antenna spacings and angle spreads at the transmitter and receiver are large. We consider correlations in frequency, but assume spatial independence. The non-zero eigenvalues of $\mathbf{H}_k \mathbf{H}_k^\dagger$ are denoted by $\lambda_1^{(k)} > \lambda_2^{(k)} > \dots > \lambda_m^{(k)}$ where $m = \min(N_R, N_T)$, and the maximum eigenvalue is denoted by $\lambda_{\max}^{(k)} = \lambda_1^{(k)}$. The CDF of $\lambda_{\max}^{(k)}$ is known [4] and is denoted by $F(x) = \text{Prob}(\lambda_{\max}^{(k)} \leq x)$. For a generic subcarrier, we omit the superscript and write λ_{\max} . We will also require the notation $n = \max(N_R, N_T)$ in later sections.

B. Adaptive Modulation System

Adaptive modulation is a technique that increases the spectral efficiency under changing channel conditions. In more favorable channel conditions, a higher number of bits per symbol can be transmitted, while in less favorable conditions, modulation is downgraded to a less spectrally-efficient constellation. It is generally assumed that in adaptive modulation the system attempts to maintain a constant target bit error rate (BER) while maximizing the spectral efficiency. The CSI is fed back from the receiver to the transmitter, and the maximum eigenvalue for each subcarrier is compared with a set of fixed thresholds $\{T_1, T_2, \dots, T_{L+1}\}$, where L is the number of alternative modulation modes. In this paper, the feedback channel is assumed to be ideal (error free). If the maximum eigenvalue lies between thresholds T_i and T_{i+1} , then the i -th modulation mode is used by the transmitter. The thresholds are calculated by combining the available information regarding the channel fading model, the target BER and the spectral efficiency of the various possible modulation modes.

Here, we use an adaptive modulation scheme in which the estimated subcarrier SNR values (via the maximum eigenvalues) are used to adjust the modulation scheme. We ignore any guard interval or cyclic prefix in the OFDM block. Furthermore, we consider seven modulation options: outage, BPSK, QPSK, 8-PSK, 16-QAM, 32-QAM and 64-QAM. The SNR boundaries for switching between the modulation schemes are obtained using the approximate method for M-PSK and square M-QAM presented in [6], which are valid for received signals of the form given in (4). For a target BER of p_e these approximations are given by

$$\text{SNR}_{\text{MPSK}} \approx -\frac{1}{8} \ln(4 p_e) 2^{1.94 \frac{\ln(M)}{\ln(2)}} \quad (9)$$

$$\text{SNR}_{\text{MQAM}} \approx -\frac{2}{3} (M - 1) \ln(5 p_e). \quad (10)$$

TABLE I
MAXIMUM EIGENVALUE THRESHOLDS FOR TWO TARGET BER VALUES.

| Modulations | Maximum eigenvalue thresholds | |
|-------------|---------------------------------------|--------------------------------------|
| | Target BER = 10^{-3} | Target BER = 10^{-2} |
| Outage | $0 \leq \lambda_{\max} < 0.331$ | $0 \leq \lambda_{\max} < 0.194$ |
| BPSK | $0.331 \leq \lambda_{\max} < 1.270$ | $0.194 \leq \lambda_{\max} < 0.745$ |
| QPSK | $1.270 \leq \lambda_{\max} < 4.873$ | $0.745 \leq \lambda_{\max} < 2.861$ |
| 8-PSK | $4.873 \leq \lambda_{\max} < 6.622$ | $2.861 \leq \lambda_{\max} < 3.771$ |
| 16-QAM | $6.622 \leq \lambda_{\max} < 13.785$ | $3.771 \leq \lambda_{\max} < 7.794$ |
| 32-QAM | $13.785 \leq \lambda_{\max} < 28.014$ | $7.794 \leq \lambda_{\max} < 15.840$ |
| 64-QAM | $28.014 \leq \lambda_{\max}$ | $15.840 \leq \lambda_{\max}$ |

Using (3), we can obtain the modulation switching thresholds from the following expression

$$\frac{E_s}{\sigma^2} \lambda_{\max} = \text{SNR}_{\text{MPSK}}(\text{SNR}_{\text{MQAM}}). \quad (11)$$

Substituting (9) or (10) into (11) gives threshold values for λ_{\max} which can be used to implement the adaptive modulation scheme. The threshold values are summarized in Table I, where we have assumed an average SNR per branch equal to 9dB and target BER values of 10^{-2} and 10^{-3} .

C. Performance Metrics

Most existing work on OFDM focuses on the mean performance and relies on results for a single subcarrier, which are usually straightforward. For example, the mean SER of our system is simply $\text{SER} = \text{Prob}(R_k \text{ is not decoded as } S_k)$, and the outage of each subcarrier is $\text{Prob}(\lambda_{\max}^{(k)} < T)$, where T is some threshold below which the channel is deemed to be in outage. These results are identical for every subcarrier, and such metrics give mean results with no indication as to the behavior of the whole block. In this paper we consider block-based metrics such as the number of outages in the block and the data rate of the block. Extensions to other metrics such as BER and capacity [1] may also be possible. Consider the binary-valued function

$$B_k = \begin{cases} 0, & \text{if } 0 \leq \lambda_{\max}^{(k)} < T \\ 1, & \text{if } T \leq \lambda_{\max}^{(k)} < \infty \end{cases} \quad (12)$$

where T is the threshold value below which modulation is suspended, i.e., an outage occurs. The function B_k simply counts whether the k -th bin is ON or OFF. Also, consider the more general function

$$W_k = \begin{cases} w_1, & \text{if } T_1 \leq \lambda_{\max}^{(k)} < T_2 \\ w_2, & \text{if } T_2 \leq \lambda_{\max}^{(k)} < T_3 \\ \vdots & \\ w_L, & \text{if } T_L \leq \lambda_{\max}^{(k)} < T_{L+1} \end{cases} \quad (13)$$

which includes any metric that measures a fixed criterion based on $\lambda_{\max}^{(k)}$ in each bin. If w_i is the number of bits used in the i -th modulation scheme, then $W = \sum_{k=1}^N W_k$ counts the total number of bits transmitted per OFDM block and $B = \sum_{k=1}^N B_k$ gives the total number of times the modulation is ON per OFDM block. Since the number of outages in the

block is $N - B$, we note that B gives outage information. Similarly, the data rate of the block is proportional to W so W gives the data rate information. Since B is a special case of W , we consider only W in the following analysis.

III. MIMO OFDM SYSTEM ANALYSIS

As discussed in [7], the exact distribution of W is prohibitively complex due to the difficulties in using the joint PDF of $(\lambda_{\max}^{(1)}, \lambda_{\max}^{(2)}, \dots, \lambda_{\max}^{(N)})$. Alternatively, since W is a sum of random variables, for a large number of subcarriers we might suppose that the distribution of W is approximately Gaussian, based on some variation of the central limit theorem (CLT). However, (6) shows that the correlations $\text{E}[X_{k_1} Y_{k_2}]$ decay with order $\frac{1}{\Delta k}$ as the separation in frequency increases. This is a strongly correlated scenario, and ordinary CLT arguments for correlated variables may not be valid. Hence, we use a theorem due to Arcones [8] previously adapted for use in OFDM research in [2]. The work in [2] was for SISO OFDM systems, but it is straightforward to extend it to the MIMO case.

We state the Arcones theorem below, which applies to the case where the number of subcarriers N increases and Δf remains fixed. Hence, we have a CLT for the case of increasing bandwidth. Note that, as the bandwidth increases, for fixed Δf and E_s the total power will also increase. Thus, the CLT assumes that the power increases indefinitely as the number of subcarriers increases. Although this is unrealistic, the main purpose of the CLT is to validate the use of a Gaussian approximation for the finite bandwidth case, and here the problem of increasing power is not an issue.

Theorem 1 (Arcones-de Naranjo) Let $\{\mathbf{X}_j\}_{j=1}^\infty$ be a stationary mean-zero sequence of Gaussian vectors in \mathbb{R}^d . Set $\mathbf{X}_j = (X_{j,1}, \dots, X_{j,d})$. Let g be a function on \mathbb{R}^d with Hermite rank $\varphi(g)$ such that $1 \leq \varphi(g) < \infty$. Define

$$r^{(p,q)}(k) = \text{E}[X_{m,p} X_{m+k,q}] \quad (14)$$

for $k \in \mathbb{Z}$, where m is any number large enough that $m \geq 1$ and $m+k \geq 1$. Suppose that

$$\sum_{k=-\infty}^{\infty} |r^{(p,q)}(k)|^{\varphi(g)} < \infty, \quad (15)$$

for all $1 \leq p \leq d$ and $1 \leq q \leq d$. Then, as $N \rightarrow \infty$,

$$\frac{1}{\sqrt{N}} \sum_{j=1}^N \{g(\mathbf{X}_j) - \text{E}[g(\mathbf{X}_j)]\} \xrightarrow{D} \mathcal{N}(0, \sigma_g^2) \quad (16)$$

where ' \xrightarrow{D} ' denotes 'convergence in distribution'.

We apply this theorem to the case where $\mathbf{X}_j = \text{vec}(\mathbf{H}_j)$, $d = 2N_R N_T$, N is the number of subcarriers and $g(\mathbf{X}_j) = g(\text{vec}(\mathbf{H}_j)) = W_j$. From (6), condition (15) is simple to verify as long as the Hermite rank of g is at least two [2]. In [7] we demonstrate that the Hermite rank is at least two, which is intuitively sensible since $g(\mathbf{X}_j)$ is a symmetric function of \mathbf{X}_j . Hence, the theorem supplies a CLT for W .

The convergence in distribution described in (16) clearly motivates the following approximation. For large finite N the distribution of W may be approximated by a Gaussian random variable with mean $E[W] = N E[W_i]$ and variance

$$\text{var}[W] = N \text{var}[W_i] + 2 \sum_{k=1}^{N-1} (N-k) \text{cov}[W_1, W_{1+k}] \quad (17)$$

Since we are using a CLT for W , the approximate distribution depends solely on $E[W]$ and $\text{var}[W]$. But, we have mean $E[W] = N E[W_i]$, where

$$E[W_i] = \sum_{k=1}^L w_k [F(T_{k+1}) - F(T_k)] \quad (18)$$

and the variance is given by (17), where

$$\text{var}[W_i] = \sum_{k=1}^L w_k^2 [F(T_{k+1}) - F(T_k)] - E[W_i]^2 \quad (19)$$

$$\begin{aligned} \text{cov}[W_1, W_{1+k}] &= E[W_1 W_{1+k}] - E[W_1]^2 \\ &= \sum_{i=1}^L \sum_{j=1}^L w_i w_j \text{Prob}(T_i \leq \lambda_{\max}^{(1)} < T_{i+1}, \\ &\quad T_j \leq \lambda_{\max}^{(k+1)} < T_{j+1}) - E[W_1]^2. \end{aligned} \quad (20)$$

These equations can be evaluated if we know the marginal and joint probabilities of the maximum eigenvalues in bins 1 and $k+1$.

A. Derivation of the Joint Cumulative Distribution Function

To complete the calculation of (17) – (20), we require the joint CDF of $\lambda_{\max}^{(1)}$ and $\lambda_{\max}^{(k+1)}$ defined by $F_k(x, y) = \text{Prob}(\lambda_{\max}^{(1)} \leq x, \lambda_{\max}^{(k+1)} \leq y)$. Note that the marginal CDFs can be obtained directly from the joint CDF.

The calculation of the joint CDF of $\lambda_{\max}^{(1)}$ and $\lambda_{\max}^{(k+1)}$ relies on a result in [9], where the joint PDF of the ordered eigenvalues $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m) = (\lambda_1^{(1)}, \dots, \lambda_m^{(1)})$ and $\boldsymbol{w} = (w_1, \dots, w_m) = (\lambda_1^{(k+1)}, \dots, \lambda_m^{(k+1)})$ is shown to be

$$\begin{aligned} f_o(\boldsymbol{w}, \boldsymbol{\lambda}) &= C_o (1 - \rho^2)^{-m} \rho^{-m(n-1)} \\ &\times \exp \left\{ -\frac{1}{1 - \rho^2} \sum_{k=1}^m (w_k + \lambda_k) \right\} \prod_{i < j} [(\lambda_i - \lambda_j)(w_i - w_j)] \\ &\times \left| (\lambda_i w_j)^{(n-m)/2} I_{n-m} \left(2\sqrt{\mu \lambda_i w_j} \right) \right| \end{aligned} \quad (21)$$

where $\rho = |\rho_f(\Delta k \Delta f)|$, $\mu = \rho^2(1 - \rho^2)^{-2}$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$, $w_1 \geq w_2 \geq \dots \geq w_m$ and

$$C_o \triangleq \{ \prod_{k=1}^m [(n-k)! (m-k)!] \}^{-1}. \quad (22)$$

The notation $|M_{ij}|$ refers to the determinant of an $m \times m$ matrix \boldsymbol{M} with (i, j) -th element M_{ij} . Note the slight abuse of notation where, for convenience, we have rewritten $\lambda_i^{(1)}$ and $\lambda_i^{(k+1)}$ as λ_i and w_i respectively. Using some determinant results in [10], we are able to integrate out $(\lambda_2, \dots, \lambda_m)$ and (w_2, \dots, w_m) from (21) to obtain the required CDF

$$F_k(x, y) = K |A_{ij}(x, y)| \quad (23)$$

where $K = (1 - \rho^2)^{-m} \rho^{-m(n-1)} C_o$ and $|A_{ij}(x, y)|$ represents the determinant of the $m \times m$ matrix $\boldsymbol{A}(x, y)$ with (i, j) -th element

$$\begin{aligned} A_{ij}(x, y) &= \\ &\sum_{k=0}^{\infty} \frac{\mu^{k + \frac{n-m}{2}} \gamma(n-m+j+k, \delta y) \gamma(n-m+i+k, \delta x)}{\delta^{n-m+j+k} \delta^{n-m+i+k} k! (k+n-m)!}. \end{aligned} \quad (24)$$

In (24), $\gamma(\cdot, \cdot)$ is the incomplete gamma function $\gamma(\alpha, \beta) = \int_0^\beta t^{\alpha-1} e^{-t} dt$ and δ is defined by $\delta = (1 - \rho^2)^{-1}$. Brief details of the derivation can be found in the Appendix.

Hence, the joint and marginal distributions can be found by computing $A_{ij}(x, y)$. Although the infinite series for $A_{ij}(x, y)$ is not desirable, we have found that the series converges quickly. For example, when we consider a (4,4) MIMO OFDM system with $N = 64$ and $\tau_d = 100\text{ns}$, the series in (24) converges in less than 45 iterations to within typical machine accuracy. For larger values of τ_d or smaller system sizes, the convergence is even faster. The marginal probabilities can then be computed by using $\text{Prob}(T_i \leq \lambda_{\max} < T_{i+1}) = F_k(T_{i+1}, \infty) - F_k(T_i, \infty)$. Utilizing the joint and marginal probabilities, the mean and variance can be readily obtained. In Sec. V we compare our analytical results using this Gaussian fit method with simulation data and then use them to study the effects of the various system parameters.

IV. LCR AND AFB

In this section our focus moves to another block-based perspective. Here we are interested in how the channel gain fluctuates over the bandwidth of the OFDM system. To be specific, we aim to derive analytical expressions for the LCR and AFB in the frequency domain (written as LCR_f and AFB_f , respectively) for the subcarrier gains, including the link gain of SISO OFDM systems and the eigenmode gain of MIMO OFDM systems using MRT-MRC.

A. Subcarrier Link Gain in SISO OFDM

It is well known that the link gain of a SISO Rayleigh fading channel, $|H|^2$, is a complex chi-squared (χ^2) process with one degree-of-freedom, as the gain is the sum of squares of two i.i.d. Gaussian components (the real and imaginary parts in (5)). Hence, following [11], the corresponding LCR for the process in the frequency domain is given by

$$\text{LCR}_{f, |H|^2}(T) = \sqrt{\frac{-\ddot{\rho}_f(0) T}{\pi}} \exp(-T) \quad (25)$$

where T is the threshold level and $\ddot{\rho}_f(0)$ is the second derivative of the correlation function of the underlying Gaussian process at $\Delta f = 0$.

From (7) the correlation function of the underlying Gaussian process with frequency separation Δf can be expanded as

$$\rho_f(\Delta f) \approx 1 + j 2\pi \tau_d \Delta f - \frac{(2\pi \tau_d \Delta f)^2}{2}. \quad (26)$$

Hence, the curvature of $\rho_f(\Delta f)$ at $\Delta f = 0$ is given by $\ddot{\rho}_f(0) = -4\pi^2\tau_d^2$. This gives the very simple closed-form LCR formula for $|H|^2$ in the frequency domain given by

$$\text{LCR}_{f,|H|^2}(T) = 2\tau_d\sqrt{\pi T}\exp(-T). \quad (27)$$

Clearly, LCR_f is proportional to τ_d . Additionally, the AFB can be computed using the well-known relationship between the LCR and AFB

$$\text{AFB}_{f,|H|^2}(T) = \frac{\text{Prob}(|H|^2 < T)}{\text{LCR}_{f,|H|^2}(T)} = \frac{1 - e^{-T}}{2\tau_d\sqrt{\pi T}e^{-T}}. \quad (28)$$

The formulas (27) and (28) are derived assuming that the subcarrier gain is a continuous process in frequency. The simulations, however, consider a discrete process over the N frequencies, f_1, f_2, \dots, f_N . For small τ_d values such as 100ns, this difference is not important, as the process is very smooth ($|\rho_f(\Delta f)| \approx 0.9813$) and the continuous approximation is very accurate. Increasing the value of $\Delta f\tau_d$ results in a lower $\rho_f(\Delta f)$, and the process tends to become more discrete. This leads to reduced accuracy in the formulas (27) and (28). We propose a method to ameliorate this problem in [7].

B. Subcarrier SNR in MIMO OFDM

In this paper we are interested in how the maximum eigenvalues and hence the subcarrier SNR values evolve with frequency in a MIMO OFDM channel. The analysis of the LCR and average fade duration (AFD) of MIMO eigenmodes over time has been investigated in [12], [13], [14]. In particular, a very simple method for LCR computation has been given in [14], and the application of this technique is extended here to derive the LCR and AFB for MIMO eigenmodes in the frequency domain.

As shown in [13], [14], the eigenvalues as well as the singular values $s = \sqrt{\lambda}$ can be accurately approximated by gamma processes. As a result, the LCR for the eigenvalue process can be approximated using

$$\text{LCR}_{f,\lambda}(T) = \frac{1}{2\Gamma(r)} \sqrt{\frac{2|\ddot{R}_s(0)|}{\pi}} \left(\theta\sqrt{T}\right)^{r-0.5} e^{-\theta\sqrt{T}} \quad (29)$$

where $r = \{E[s]\}^2/\text{var}[s]$ and $\theta = E[s]/\text{var}[s]$ are the shape and scale factors of the gamma variable that approximates the singular value process. Note that these parameters depend solely on the first two moments of the singular value process, and hence can be acquired from the distribution of the eigenvalues. More details on computing $E[s]$ and $\text{var}[s]$ can be found in [15]. Also, following the same argument as in [14], $\ddot{R}_s(0)$ is the curvature of the correlation function of the singular value s , which is given by

$$\ddot{R}_s(0) = \frac{2\pi^2\tau_d^2\theta^2}{r}. \quad (30)$$

Hence, we have the closed-form LCR formula

$$\text{LCR}_{f,\lambda}(T) = \sqrt{\frac{\pi}{r}} \frac{\tau_d\theta}{\Gamma(r)} \left(\theta\sqrt{T}\right)^{r-0.5} e^{-\theta\sqrt{T}}. \quad (31)$$

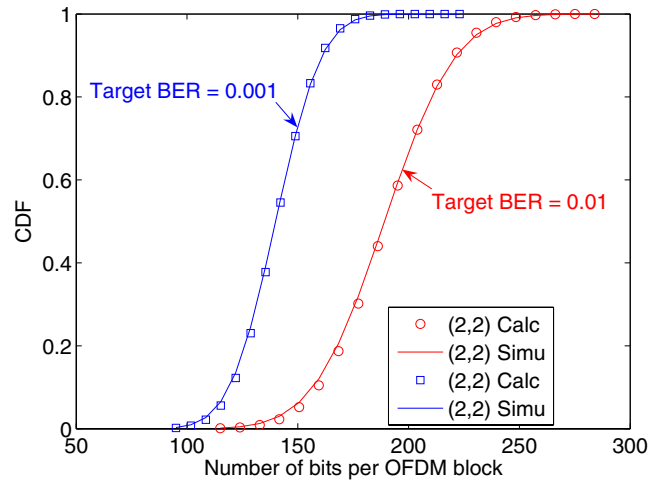


Fig. 1. CDF plots for number of bits per OFDM block in a (2,2) MIMO OFDM for two different target BERs ($N = 64$, $\tau_d = 100\text{ns}$).

From (31), we can conclude that the LCR for the eigenmode in the frequency domain is also proportional to τ_d . This formula simply requires the first two moments (for r , θ) of the corresponding singular value process, both of which can be acquired from the Wishart distribution [15]. Moreover, the AFB for the eigenmode gain is easily computed using

$$\text{AFB}_{f,\lambda}(T) = \frac{\text{Prob}(\lambda < T)}{\text{LCR}_{f,\lambda}(T)} \quad (32)$$

where $\text{Prob}(\lambda < T)$ can be calculated using either its gamma approximation or the exact marginal density of the eigenvalue [15].

V. SIMULATION RESULTS

The simulations were carried out for a 64 subcarrier system with subcarrier separation $\Delta f = 0.3125\text{MHz}$, thus occupying a bandwidth of 20MHz. Although 64 subcarriers is relatively small for practical systems, this number was chosen to demonstrate the accuracy of the CLT for a small number of carriers. Also, a system carrier frequency of 5.725GHz (HyperLan 2 standard) was chosen. In our first set of results we compare the Gaussian CDF based on the CLT using an analytically-derived mean and variance with subcarrier modulation statistics obtained from Monte Carlo simulations. We evaluate two systems with target BERs of 10^{-3} and 10^{-2} , respectively. Furthermore, our simulations were carried out to observe the effect of correlation across frequency on the approximating distributions. We consider mean delay spreads of 100ns and 250ns, which give correlation coefficients of $|\rho_f(\Delta f)| = 0.9813$ and 0.8977 respectively, for an exponential power delay profile. Figures 1 – 3 show excellent agreement between the Gaussian approximation and simulated data rates for OFDM blocks. Note that in Fig. 1 we have only 64 subcarriers, and the correlation between adjacent subcarriers is of magnitude 0.9813. Hence, the CLT is worked quite hard but still yields excellent results.

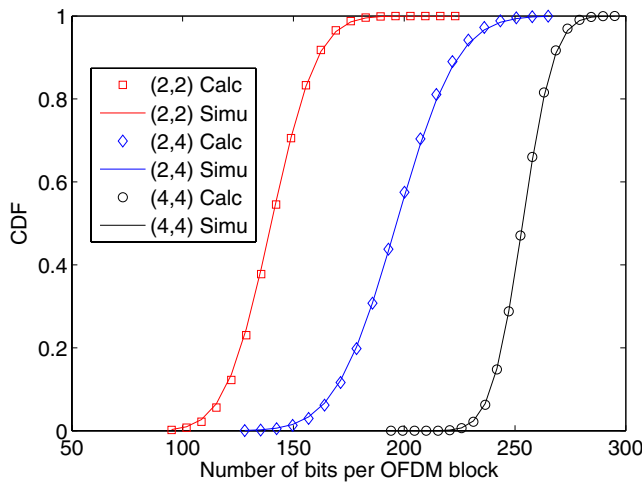


Fig. 2. CDF comparison between analysis and simulation for various MIMO system configurations ($N = 64$, $\tau_d = 100\text{ns}$, Target BER = 10^{-3}).

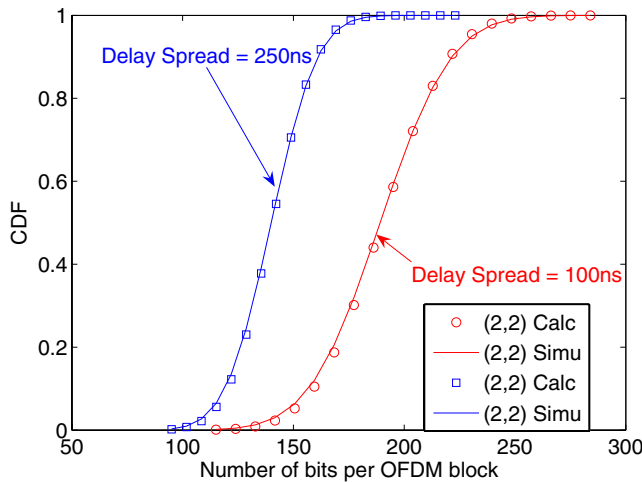


Fig. 3. CDF comparison between analysis and simulation for two different delay spreads τ_d ($N = 64$, Target BER = 10^{-3}).

In our second set of simulations we examine the LCR and AFB for MIMO channel gains. The LCR formula for the eigenmode in the frequency domain has been derived in (31). The accuracy of this formula for MIMO systems with different sizes is exhibited in Fig. 4. Note that, although we are particularly interested in the largest eigenvalue in this paper, our formula is valid for any eigenvalues of interest. We plot the AFB for the largest eigenvalue in a (2, 2) MIMO OFDM system in Fig. 5. The simulated AFB saturates above a certain threshold level due to the limited bandwidth of the OFDM system. Nevertheless, we see that for the moderate to low thresholds which are of most interest the analytical AFB is very accurate.

VI. CONCLUDING REMARKS

In this paper we have considered some fundamental issues concerning the performance of adaptive MIMO OFDM systems and the behavior of the channel across frequency.

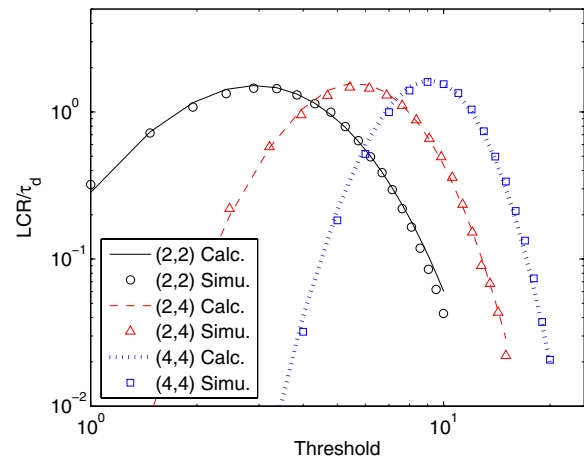


Fig. 4. Comparison between simulated and calculated normalized LCR for largest eigenmode in MIMO OFDM systems with different sizes ($N = 64$, $\tau_d = 100\text{ns}$).

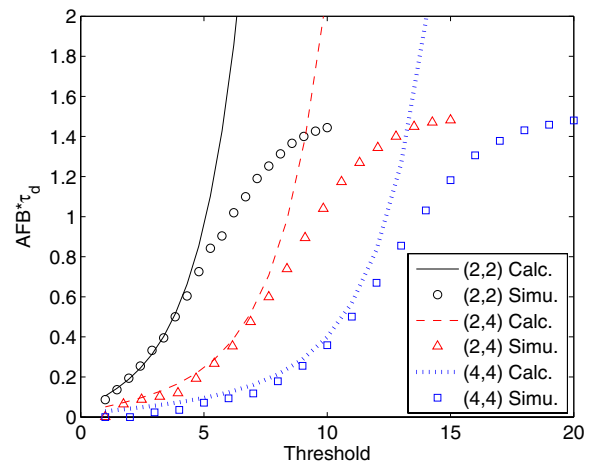


Fig. 5. Comparison between simulated and calculated AFB for the largest eigenmode of a (2,2) MIMO OFDM system ($N = 64$, $\tau_d = 100\text{ns}$).

Focusing on wideband channel variations in the frequency domain, we have considered data rate metrics and derived exact results for their means and variances. Furthermore, a CLT was developed, and the resulting Gaussian approximation has shown excellent agreement with our simulated results. We have also derived very accurate approximations to the LCRs and AFB of the MIMO eigenmodes. Our approximations can be further used to predict the frequency-varying characteristics of the MIMO channel in limited feedback systems where the channel information for only a subset of subcarriers is fed back to the transmitter.

APPENDIX

The joint density function of λ and w is given in (21) and can be rewritten as

$$f_o(w, \lambda) = K e^{-\sum_{i=1}^m \delta w_i} e^{-\sum_{i=1}^m \delta \lambda_i} |V(\lambda)| |V(w)| \times \left| (\lambda_i w_j)^{\frac{n-m}{2}} I_{n-m} \left(2\sqrt{\mu \lambda_i w_j} \right) \right| \quad (33)$$

where $V(\cdot)$ represents the Vandemonde matrix with $V(\lambda) = (\lambda_j^{i-1})$ and determinant $|V(\lambda)| = \prod_{i < j} (\lambda_i - \lambda_j)$. We now apply a result in Corollary 2 of Chiani *et al.* [10] which states that

$$\int_S |\Phi(x)| |\Psi(x)| \prod_{k=1}^m \xi(x_k) dx = \left| \int_a^b \Phi_i(x) \Psi_j(x) \xi(x) dx \right| \quad (34)$$

where $x = (x_1, x_2, \dots, x_m)$, \int_S represents m -dimensional integration over the region $b \geq x_1 \geq \dots \geq x_m \geq a$ and $\Phi(x)$, $\Psi(x)$ are $m \times m$ matrices with (i, j) -th elements of the form $\Phi_i(x_j)$ and $\Psi_j(x_i)$, respectively. We apply this result with $x = w$, $\Phi(w) = V(w)$, $\Psi(w) = \left([\lambda_i w_j]^{\frac{n-m}{2}} I_{n-m} [2\sqrt{\mu \lambda_i w_j}] \right)$ and $\xi(w_i) = e^{-\delta w_i}$. The result is given by

$$\int_S f_o(w, \lambda) dw = K e^{-\sum_{i=1}^m \delta \lambda_i} |V(\lambda)| \times \left| \int_a^b w^{i-1} (\lambda_j w)^{\frac{n-m}{2}} I_{n-m} (2\sqrt{\mu \lambda_j w}) e^{-\delta w} dw \right|.$$

Now we apply (34) again with $x = \lambda$, $\Phi(\lambda) = V(\lambda)$,

$$\Psi(\lambda) = \left(\int_a^b w^{i-1} (\lambda_j w)^{\frac{n-m}{2}} I_{n-m} (2\sqrt{\mu \lambda_j w}) e^{-\delta w} dw \right)$$

and $\xi(\lambda_i) = e^{-\delta \lambda_i}$. Also, the m -dimensional integral, denoted \int_S , is replaced by \int_T where \int_T denotes integration over the region $d \geq \lambda_1 \geq \dots \geq \lambda_m \geq c$. This gives

$$\int_T \int_S f_o(w, \lambda) dw d\lambda = K \left| \int_c^d \lambda^{i-1} \int_a^b w^{j-1} (\lambda w)^{\frac{n-m}{2}} \times I_{n-m} (2\sqrt{\mu \lambda w}) e^{-\delta w} dw e^{-\delta \lambda} d\lambda \right|. \quad (35)$$

To compute (35) we require integrals of the form

$$\int_c^d \left[\int_a^b w^{\frac{n-m}{2} + j - 1} e^{-\delta w} I_{n-m} (2\sqrt{\mu \lambda w}) dw \right] \times \lambda^{\frac{n-m}{2} + i - 1} e^{-\delta \lambda} d\lambda. \quad (36)$$

Using the series expansion for the modified Bessel function,

$$I_n(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{x^2}{4}\right)^{k+\frac{n}{2}}}{k!(k+n)!},$$

equation (36) can be rewritten as

$$\sum_{k=0}^{\infty} \frac{\mu^{k+\frac{n-m}{2}}}{k!(k+n-m)!} \left[\int_a^b w^{n-m+j+k-1} e^{-\delta w} dw \right] \times \left[\int_c^d \lambda^{n-m+i+k-1} e^{-\delta \lambda} d\lambda \right]. \quad (37)$$

Setting $a = c = 0$ in (35) and using (37) gives

$$\text{Prob}(\lambda_1 \leq d, w_1 \leq b) =$$

$$K \left| \sum_{k=0}^{\infty} \frac{\mu^{k+\frac{n-m}{2}} \gamma(n-m+j+k, \delta b) \gamma(n-m+j+k, \delta d)}{\delta^{n-m+j+k} \delta^{n-m+i+k} k!(k+n-m)!} \right| \quad (38)$$

where the integrals in (37) are expressed in terms of incomplete gamma functions. This gives the desired result in (23).

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